SP222 Electricity and Magnetism I
LRC Circuits

Purpose   To study the behavior of a series LRC circuit without a generator, and compare the results with the theoretical description of a damped harmonic oscillator.

Reference   Tipler, Physics (4th edition), pp. 908-972.

Introduction   A series *LRC* circuit consists of an inductor *L*, a resistor *R*, and a capacitor *C* connected together as shown in Figure 1. Suppose that the switch *S* is first placed in the position labeled 1. The battery will charge the capacitor. If the switch is then moved to the position labeled 2, the charge on one plate of the capacitor will flow to the other, through the resistor and the inductor. If the resistance in the circuit is not too large, the capacitor will become charged again as a result of this flow, but the magnitude of the charge on each plate will be smaller than it was originally, and the polarity of the capacitor will be reversed. The process of discharging and charging will then repeat itself, with the charge on the capacitor following a series of damped oscillations.

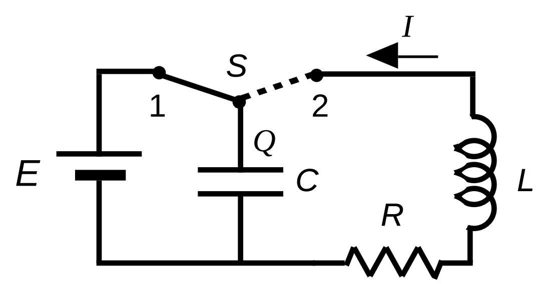

Figure 1. Series LRC Circuit

Applying the loop rule to this circuit gives

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = 0, \quad \text{with} \quad I = \frac{dQ}{dt}$$

so

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = 0 \quad \text{or} \quad \frac{d^2Q}{dt^2} + 2b\frac{dQ}{dt} + \omega_0^2 Q = 0 \qquad (1)$$

where $b = R/(2L)$. Assuming a solution of the form $Q = Be^{\gamma t}$, we substitute into Equation (1) to obtain $\gamma^2 + 2b\gamma + \omega_0^2 = 0$. Solving this equation for γ gives

$$\gamma = \frac{-2b \pm \sqrt{4b^2 - 4\omega_0^2}}{2} = -b \pm \sqrt{b^2 - \omega_0^2}$$

There are three possible types of answer here depending on the relative size of b^2 and ω_0^2, and there are three special names given to the corresponding types of motion. We say the motion is

overdamped if	$b^2 > \omega_0^2$
critically damped if	$b^2 = \omega_0^2$
underdamped or oscillatory if	$b^2 < \omega_0^2$

Overdamped Motion Since $\sqrt{b^2 - \omega_0^2}$ is real and less than b, both roots of Equation (2) are negative, and the general solution is a linear combination of two negative exponentials:

$$Q = Ae^{-\left(b+\sqrt{b^2-\omega_0^2}\right)t} + Be^{-\left(b-\sqrt{b^2-\omega_0^2}\right)t}$$

Critically Damped Motion Since $b = \omega_0$, the two roots of Equation (2) are equal. In this case the solution to Equation (1) is given by

$$Q = (A + Bt)e^{-bt}$$

Underdamped (Oscillatory) Motion Since $b^2 < \omega_0^2$, $\sqrt{b^2 - \omega_0^2}$ is imaginary, so $\sqrt{b^2 - \omega_0^2} = i\sqrt{\omega_0^2 - b^2}$. Let $\omega' = \sqrt{\omega_0^2 - b^2}$. Then the roots of Equation (2) are $\lambda = b \pm i\omega'$ and the solution to Equation (1) is

$$\begin{aligned} Q &= Ae^{(-b+i\omega')t} + Be^{(-b-i\omega')t} \\ &= Ae^{-bt}(\cos\omega't + i\sin\omega't) + Be^{-bt}(\cos\omega't - i\sin\omega't) \\ &= (A+B)e^{-bt}\cos\omega't + (A-B)ie^{-bt}\sin\omega't \end{aligned}$$

For initial (t = 0) conditions Q = Q_0 and I = dQ/dt = 0 we obtain $A + B = Q_0$ and $A - B = 0$. Thus,

$$Q = Q_0 e^{-bt}\cos\omega't$$

In today's experiment, you will attempt to verify the descriptions of critically damped and underdamped motion. As in previous experiments, you will not be able to measure the charge on the capacitor directly. Instead you will measure the potential difference V across it, and deduce the charge using the relationship $Q = CV$. If the charge oscillates, so will the potential difference, so the voltage across the capacitor in the underdamped *LRC* circuit will be given by

$$V = V_0 e^{-bt}\cos\omega't \qquad (2)$$

Procedure

Part I. Setup

(1) Verify that your computer is displaying the USNA Physics Laboratories screen. Click on the icon labeled Two Chan Plot&Read. After about a minute, the startup screen will appear.

(2) Using the controls at the upper left-hand corner of the screen, select a negative trigger slope and a trigger level of about 5 V. Also, set the voltage range for Channel #1 to -10 to +10 V, the sampling rate to about 10 kHz, and the number of samples per channel to about 300.

(3) Open Data.Editor. Then you can use the pull-down Window menu to switch back and forth between Two Chan Plot&Read and Data.Editor whenever you wish. Switch back to Two Chan Plot&Read for now.

(4) Wire up the circuit shown in Figure 2, on the last page of this write-up. Note that Channel 1 is connected across the capacitor. Short and ground the input of Channel 2. Set the power supply output to 10 V.

Part II. Oscillation frequency for an underdamped circuit

(1) Set the resistance substitution box to R = 0 Ω and the capacitance box to C = 1.0 µF. Place the switch in position 1, charging the capacitor. Click on the button labeled Read/Plot, and quickly but smoothly move the switch to position 2. After a moment, several cycles of the measured capacitor voltage should appear on the Two Chan Plot&Read screen. If the software does not trigger, check your connections and the trigger settings and try again.

(2) Make a printout of your plot. From the printout, find the period of oscillation, and, using that, compute the *measured value* angular frequency ω'. (For better results, you can save your data to a file and then examine it with either Excel or Data.Editor to find the oscillation period.) Use the equation $\omega' = \sqrt{\omega_0^2 - b^2}$ to calculate the *expected value* for ω', and compare it with the *measured* one. **Note**: The resistance of the inductor is not negligible, and must be taken into account in all of your calculations. You will find a value for the resistance written on the inductor.

(3) Repeat these measurements and calculations, keeping the resistance substitution box set for R = 0 Ω and using C = 0.25 µF and 4.0 µF. Adjust the sampling rate, if necessary.

Part III. Damping rate for an underdamped circuit

(1) Adjust either the capacitance or the number of samples per channel so that you can see several cycles of the damped oscillation. Make a printout of your screen. Save the data in a file.

(2) Use Data.Editor to find the amplitudes of several successive peaks and the times at which they occur. Make a file containing your peak-amplitude vs. time data. Each line of the file should contain the amplitude and time of occurrence for one peak.

(3) Make a plot of the natural log of the peak amplitude vs. time. Fit a straight line to the graph, and find the slope. Use Equation (2) to relate b with the slope and use this relation to find the *measured value* of b. Use the equation $b = R/(2L)$ to calculate the *expected value* of b, and compare it with the *measured* one.

Part IV. Resistance for a critically damped circuit

(1) Set C = 1.0 µF and calculate the value of the resistance necessary for critical damping. Set the resistance substitution box so that the total resistance in the circuit is equal to the value you calculated. Don't forget the resistance of the inductor.

(2) Make a data run with this value of resistance, and follow with a run with half this resistance and one with twice this resistance. Print the screen for each run. Comment on the appearance of the graphs, and how they demonstrate underdamped, critically damped, and overdamped behavior. If necessary, try other values of resistance to be certain that you have seen all three cases.

Part V. Effects of changing the inductance

(1) Repeat one of the cases you studied earlier (you choose), but this time place several iron rods in the open core of the inductor.

(2) Make a printout and comment on the differences (if any) that you observe from the previous results. Can you interpret the differences in terms of the changed inductance of the coil?

Figure 2. Practical Implementation of a Series LRC Circuit